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MOTION OF A VISCOUS GAS IN A LAYER WITH AN ELASTIC BOUNDARY

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The problem of axisymmetric flow of air between a flat wall and an elastic ring diaphragm is considered. Approximate Reynolds equations for the flow of an incompressible lubricant and equations of the zero-moment theory of large deflections of a plate are used. The system of two nonlinear differential equations is solved by the method of external and internal expansions. Formula for the determination of the smallest ring gap is derived. The external problem is reduced to a variational one and solved with the use of Legendre polynomials. Results of calculations are compared with data obtained by special experiments.

The results presented here may prove useful for analyzing the operation of the air-cushion type of equipment used for moving heavy objects on the shop floor. In such equipment the upper cushion boundary consists of an elastic diaphragm.

1. The aerostatic support (AS) is diagrammatically shown in Fig. 1. An elastic diaphragm is attached to the lower side of a disk at its center by a clamping plate of radius a and at the periphery by a clamping ring of inner radius b. In the undeformed state the diaphragm is pressed to the disk. The space between the bearing surface and the diaphragm, which we shall call the "air cushion", is connected to the compressed air system via a central hole and a number of holes in the diaphragm. The space between the disk face and the diaphragm, which we shall call "the container", is connected to the external compressed air system through a number of other holes. When the resulting pressure force in the air cushion exceeds the load on the AS, the equipment is lifted with its load, and the supplied air escapes into the atmosphere through the circular gap between the diaphragm and the bearing surface (see, e. g., [1]).

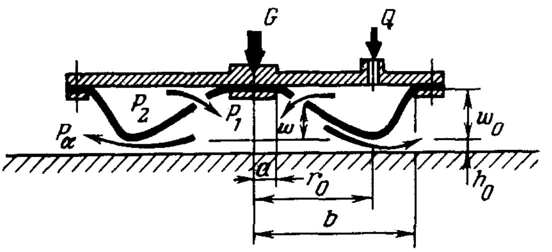

Fig. 1

The air flow in the air cushion is accompanied by formation of boundary layers on the diaphragm and on the bearing surface. Both layers merge in the neighborhood of the minimum gap which is of the order of 0.1 mm. Reynolds equations for the axisymmetric flow of an incompressible fluid may be used for determining the flow of gas in the neighborhood of the minimum gap at $r = r_0$. These equations with allowance for averaged terms of acceleration yield for the pressure drop the following equation:

$$\frac{dp}{dr} = -\frac{12\mu v}{h^2} - \rho\frac{\partial}{\partial r}\left(\frac{3}{5}v^2\right), \quad v = \frac{Q}{2\pi rh} \tag{1.1}$$

where μ is the coefficient of dynamic viscosity and v is the average velocity related to the rate of flow Q and the varying thickness h of the layer.

Since the axial displacement of points of the elastic boundary w are not small in comparison with the diaphragm thickness δ, it is necessary to resort to the theory of large deflections of plates developed by Timoshenko [2]. Below we use the simplified theory of large deflections of plates on the assumption that the effects of bending moments M_r and M_t, and of viscous forces acting on the diaphragm owing to the flow of air in the air cushion can be neglected. On these assumptions the equations of equilibrium of forces per unit of length of the meridional cross section curve of the deformed plate reduces to the single equation for the radial tension

$$\frac{1}{r}\frac{d}{dr}\left(rN_r\frac{dw}{dr}\right) = p - p_2 \tag{1.2}$$

where p is the pressure in the air cushion on the lower side of the diaphragm and p_2 is the pressure in the container on the upper side of the diaphragm.

Hooke's law and the equation of equilibrium which relate the radial and lateral tensions, yield the following differential equation:

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(r^2N_r)\right] + \frac{1}{2}\frac{E\delta}{r}\left(\frac{dw}{dr}\right)^2 = 0 \tag{1.3}$$

Equations (1.1) and (1.2) were numerically integrated in [3] on the assumption of constant radial tension throughout the air cushion, and experimental data on the diaphragm deflection and its slope at an intermediate point r $(a < r < r_0)$ were used instead of Eq. (1.3).

No experimental data are used below, and all three equations are considered, using the method similar to that of external and internal expansions. In the external expansion the thickness of the air cushion layer near the smallest circular gap, where the pressure sharply drops to atmospheric pressure p_a, is taken as the small parameter. In the first approximation the pressure is assumed to be piecewise constant, and the radius r_0 at which the pressure jump occurs is not a priori known. The diaphragm shape and the arising tensions are determined by Eqs. (1.2) and (1.3) and the fastening boundary conditions

$$w = 0 \quad \text{for} \quad r = a, \quad r = b \tag{1.4}$$

and also the condition of absence of lateral deformations at fastening boundaries

$$d/dr\,(rN_r) - \nu N_r = 0 \quad \text{for} \quad r = a, \quad r = b \tag{1.5}$$

where ν is the Poisson's ratio.

In the internal expansion the layer dimension is extended near the minimum ring gap, and it is assumed that tension N_r and raduis r vary only slightly.

2. Since the pressure in the layer sharply alters in the neighborhood of the minimum gap and the relative variation of tension N_r and radius r are small, it is possible to substitute in the first approximation in Eq. (1.2) N_0 and r_0 for these two parameters. The differentiation of Eq. (1.2), after that substitution, and the use of Eq. (1.1) yields the third order differential equation

$$\frac{d^3y}{dx^3} = \frac{1}{y^3} + A\frac{d}{dx}\frac{1}{y^2}, \quad A = \frac{\rho}{40\pi}\left[\frac{6Q^4}{h_0N_0\pi r_0^4\mu^2}\right]^{1/3} \tag{2.1}$$

$$y = \frac{h}{h_0}, \quad x = \frac{r - r_0}{\lambda}, \quad \lambda^3 = \frac{\pi r_0 N_0 h_0^4}{6\mu Q}$$

The following relation was also used:

$$h + w = H = \text{const} \approx H - h_0$$

where H is the floating platform height and h_0 is the thickness of the layer at the minimum gap circle.

For the characteristic parameters of AS the coefficient A varies within the limits $0.2 - 0.4$. Hence Eq. (2.3) can be solved by the method of successive approximations with respect to parameter A. The first approximation equation (for $A = 0$) can be reduced to a first order equation by two successive substitutions. We assume

$$dy/dx = c(y)\, y^{-1/3} \tag{2.2}$$

Then

$$\frac{d^2y}{dx^2} = cy^{-1/3}\left[y^{-1/3}\frac{dc}{dy} - \frac{1}{3}cy^{-4/3}\right] = \chi y^{-5/3} \tag{2.3}$$

$$\chi(c) = yc\frac{dc}{dy} - \frac{1}{3}c^2$$

Using (2.2) and (2.3) we obtain the following first order equation:

$$\frac{d\chi}{dc} = \frac{3 + 5c\chi}{3\chi + c^2} \tag{2.4}$$

The field of integral curves of Eq. (2.4) is shown in Fig. 2. The separating curve LM in Fig. 2 corresponds to the case in which the difference of pressure p_1 in the cushion and p_2 in the container vanishes when $x \to -\infty$. The curves above LM relate to $(p_2 - p_1) > 0$, while those below it "coil" around focus K. Such behavior of integral curves indicates that the mode $p_2 - p_1 = 0$ is not stable, since in the small neighborhood of that mode the curve of the diaphragm meridional cross section has inflection points at which $d^2y / dx^2 = 0$ or $\chi = 0$.

Asymptotics of curves lying above LM when $c \to \pm\infty$ are defined by

$$\chi = \frac{1}{2}c^2 + Bc^{4/3} + \frac{4}{5}B^2c^{-2/3} - \frac{2}{3c} \tag{2.5}$$

The asymptotics of curve LM when $c \to -\infty$ is of the form

$$\chi = -\frac{1}{2c}\left(1 + \frac{1}{6c^3} + \frac{5}{48c^6} + \cdots\right)$$

The inversion of expansion yields

$$c \approx -\frac{1}{2\chi}\left(1 - \frac{3}{4}\chi^3 + \cdots\right) \tag{2.6}$$

On the basis of notation in brackets in (2.1) we have the following conditions for function y:

$$dy/dx = 0 \quad \text{for} \quad x = 0, \quad y = 1 \tag{2.7}$$

or

$$c = 0 \quad \text{for} \quad y = 1$$

Boundaries of the considered viscous layer correspond to infinitely high values of the inner variable x. Using Eq. (1.2) and the indicated above notation, we obtain the following asymptotics:

$$\lim_{x\to-\infty}\frac{d^2y}{dx^2}=\frac{p_2-p_1}{N_0h_0}\lambda^2,\quad \lim_{x\to+\infty}\frac{d^2y}{dx^2}=\frac{p_2-p_a}{N_0h_0}\lambda^2 \tag{2.8}$$

Equation (2.1) is of the third order and there are four boundary conditions (2.7) and (2.8). These conditions are, generally speaking, incompatible, except when the coefficient $h_0N_0\lambda^{-2}$ has a specific value which depends on the ration $(p_2-p_1)(p_2-p_a)^{-1}$. The determination of this exceptional value of the indicated coefficient establishes the sought dependence of the minimum gap on parameters of the AS.

From the last equality in (2.3) we have

$$dy/dc = 3cy\,(3\chi + c^2)^{-1}$$

and taking into consideration conditions (2.7) we obtain

$$\ln y = \int_0^c 3c\,(3\chi + c^2)^{-1}\,dc \tag{2.9}$$

For the application of formula (2.9) it is necessary to have the expansion of function χ for small values of c. It is obtained by representing χ in the form of a series in positive powers of c and the determination of its coefficients by substituting it into Eq. (2.4). We have

$$\chi = \chi_0 + \frac{c}{\chi_0} + \frac{1}{2}\left(\frac{5}{3} - \frac{1}{\chi_0^3}\right)c^2 + \frac{1}{2}\left(-\frac{7}{9}\frac{1}{\chi_0^2} + \frac{1}{\chi_0^5}\right)c^3 + \dots \tag{2.10}$$

The continuation of expansion (2.6) to $c \to -0$ yields with some error $\chi_0^3 \approx 4/3$. Further calculations can be carried out with the use of equality (2.9) and asymptotics (2.5). We obtain

$$\frac{dy}{dx} \to 1,\quad \frac{d^2y}{dx^2} \approx \frac{1}{2y^2} \to 0 \quad \text{for} \quad x \to -\infty$$
$$\frac{d^2y}{dx^2} \approx 1.9\left(1 - \frac{0.075}{y^{5/2}}\right) \quad \text{for} \quad x \to +\infty \tag{2.11}$$

The limiting equalities (2.11) and Eq. (1.3) yield for the pressure drop in a viscous layer the following expression:

$$\Delta p = p_1 - p_a = 1.9h_0N_0\lambda^{-2}$$

from which we obtain

$$h_0 = 1.9\left(\frac{\mu Q}{r_0}\right)^{2/5}\left[\frac{N_0}{(\Delta p)^3}\right]^{1/5} \tag{2.12}$$

The numerical integration of Eq. (2.1) for $A = 0.3$ yields an insignificant difference: the coefficient 1.9 in formula (2.12) is replaced by 1.79.

Theoretical curves of pressure distribution (curve 2) and of the diaphragm profile near the minimum gap (curve 1) are shown in Fig. 3 by dash-dot lines. Solid lines relate to respective experimental curves. It is seen that the calculated data are in a satisfactory agreement with experimental results.

The load carrying capacity G of an aerostatic support is related to the pressure in the cushion by formula

$$G = \pi r_0^2\,(p_1 - p_a)$$

It will be shown below that tension N_0 is related to the load by formula

$$N_0 = n\left(\frac{G^2E\delta}{b^2}\right)^{1/3}$$

where the coefficient n varies between 0.15 and 0.11 when $a\,/\,b$ (the ratio of internal radius to the external one) varies from 0.2 to 0.3.

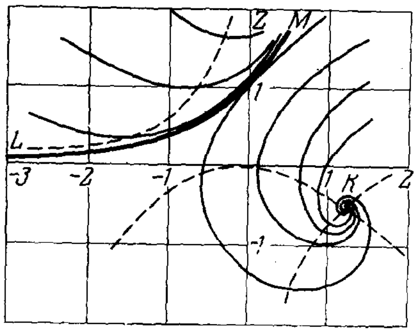

Fig. 2

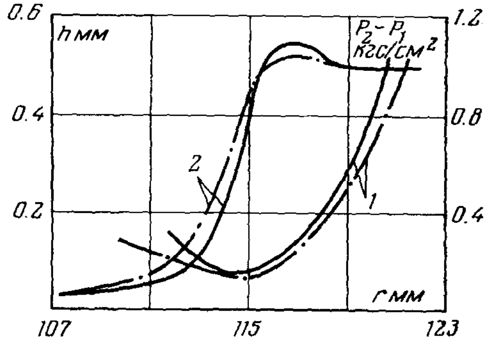

Fig. 3

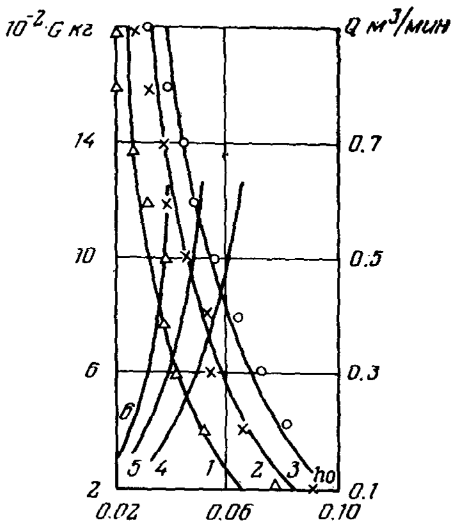

Fig. 4

The expression for the determination of the minimum gap assumes the form

$$\frac{h_0}{r_0} = 3.6\left[\frac{\mu^2 Q^4}{G^3 r_0}\left(\frac{G^2 E\delta}{b^2}\right)^{1/3}\right]^{1/4} \qquad (2.13)$$

which makes it possible to establish the possibility of using aerostatic supports for a given degree of floor roughness.

Calculated and experimental rates of flow Q (curves 4 — 6) and of load G (curves 1—3) are shown in Fig. 4 for a support with parameters $b = 302$ mm and $a/b = 0.141$.

3. The shape of the air cushion elastic boundary in the neighborhood of the minimum gap was calculated on the assumption of piecewise constant pressure which for r varying from a to r_0 equals p_1, and for the variation of r from r_0 to b is equal to atmospheric pressure p_a. It was also assumed that $p_1 = p_2$, and the effect of viscosity forces on the diaphragm was disregarded.

We introduce new variables

$$\frac{w}{\sqrt{b^2 - a^2}} = W, \quad \frac{2(r^2 - a^2)}{b^2 - a^2} - 1 = z, \quad \frac{2r^2 N_r}{E\delta(b^2 - a^2)} = N, \quad \eta = \frac{b^2}{b^2 - a^2} \qquad (3.1)$$

In these variables Eqs. (1.2) and (1.4) and boundary conditions (1.5) assume the form

$$\frac{d}{dz}\left(N\frac{dW}{dz}\right) = -Aq, \quad A = \frac{(p_2 - p_a)\sqrt{b^2 - a^2}}{8E\delta}, \quad q = \frac{p_1 - p}{p_2 - p_a} \qquad (3.2)$$

$$\frac{d^2N}{dz^2} + \left(\frac{dW}{dz}\right)^2 = 0 \qquad (3.3)$$

$$4(\eta - 1)\frac{dN}{dz} = (1 + \nu)N \quad \text{for} \quad z = -1 \qquad (3.4)$$

$$4\eta \frac{dN}{dz} = (1+\nu) N \quad \text{for} \quad z = +1 \tag{3.5}$$

We solve Eq. (3.3) for N with allowance for boundary conditions (3.4) and (3.5) and obtain

$$(1-\nu^2) N = -(1-\nu^2) \int_{-1}^{z} (z-z') \left(\frac{dW}{dz'}\right)^2 dz' + \tag{3.6}$$

$$[4(\eta-1)+(1+\nu)(1+z)] \left[\left(2\eta - \frac{1-\nu}{2}\right) \int_{-1}^{+1} \left(\frac{dW}{dz}\right)^2 dz + \right.$$

$$\left. \frac{1+\nu}{2} \int_{-1}^{+1} z \left(\frac{dW}{dz}\right)^2 dz \right]$$

By substituting for N its expression (3.6) into Eq. (3.2), we obtain for dW/dz an integral equation.

We have the following theorem: out of all possible profiles of considerable deflections of a ring diaphragm with fixed borders the functional

$$\Phi = J - A \int_{-1}^{+1} qW \, dz$$

for real deflections has a minimum, where

$$4J = \int_{-1}^{+1} \left(2\eta - 1 - \frac{1-\nu}{2} + z\frac{1+\nu}{2}\right) \left(\frac{dW}{dz}\right)^2 dz \times \tag{3.7}$$

$$\int_{-1}^{+1} \left(2\eta - 1 + \frac{1-\nu}{2} + z\frac{1+\nu}{2}\right) \left(\frac{dW}{dz}\right)^2 dz + \frac{1-\nu^2}{2} \left\{ \int_{-1}^{+1} \left[\int_{-1}^{z} \left(\frac{dW}{dz'}\right)^2 z' \, dz' \right]^2 dz - \right.$$

$$\left. \left[\int_{-1}^{+1} \left(\frac{dW}{dz}\right)^2 dz \right]^2 + \int_{-1}^{+1} \left(\frac{dW}{dz}\right)^2 dz \times \int_{-1}^{+1} z \left(\frac{dW}{dz}\right)^2 dz \right\}$$

Equating to zero the first variation of Φ for any arbitrary W with an integrable square of the derivative, this theorem yields Eq. (3.2) in which expression (3.6) is substituted for N

The indicated theorem makes it possible to obtain approximate solutions of Eqs. (3.2) and (3.3) of the form

$$\frac{dW}{dz} = \sum_{k=1}^{n} a_k f_k(z) \tag{3.8}$$

where the set of functions $f_k(z)$ is assumed orthonormal along segment $(-1, +1)$ and $f_0(z) = 2^{-1/2}$.

By virtue of this theorem the differential equations (3.2) and (3.3) are equivalent to the infinite set of algebraic equations

$$\frac{\partial J}{\partial a_k} = A \int_{-1}^{+1} q(z) f_k(z) \, dz, \quad k = 1, 2, . \tag{3.9}$$

We select the normalized Legendre polynomials

$$f_k(z) = \sqrt{\frac{2k+1}{2}} P_k(z)$$

as functions $f_k(z)$, and restrict expansion (3.8) to the first two terms. We obtain

$$\frac{dW}{dz} = \sqrt{\frac{3}{2}}\, sz + \sqrt{\frac{5}{8}}\, t\,(3z^2 - 1) \tag{3.10}$$

$$W = \frac{z^2 - 1}{2}\left(s\sqrt{\frac{3}{2}} + \sqrt{\frac{5}{2}}\, tz\right)$$

The substitution of (3.10) into (3.7) yields

$$4J = \left[\left(2\eta - 1 - \frac{1-\nu}{2}\right)(s^2 + t^2) + 2(1+\nu)\frac{st}{\sqrt{15}}\right] \times \tag{3.11}$$
$$\left[\left(2\eta - 1 + \frac{1 \mp \nu}{2}\right)(s^2 + t^2) + 2(1+\nu)\frac{st}{\sqrt{15}}\right] +$$
$$\frac{1-\nu^2}{2}\left[-\frac{3}{7}(s^2 + t^2) + \frac{4}{7}s^2t^2 + \frac{4}{231}t^4\right]$$

In conformity with (3.9) we have for the coefficients s and t the system of equations

$$\frac{\partial J}{\partial s} = -\sqrt{\frac{3}{2}}\, A \int_{-1}^{+1} q(z)\, dz \tag{3.12}$$

$$\frac{\partial J}{\partial t} = -\sqrt{\frac{5}{2}}\, A \int_{-1}^{+1} q(z)(3z^2 - 1)\, dz \tag{3.13}$$

Functions $\partial J / \partial s$ and $\partial J / \partial t$ are homogeneous third power polynomials, hence the ratio of $\partial J / \partial s$ to $\partial J / \partial t$ depends only on the ratio $t / s = \gamma$.

To determine the radius of maximum deflection z_0 we equate dW / dz to zero and obtain

$$\sqrt{\frac{3}{2}}\, z_0 + \sqrt{\frac{5}{8}}\,(3z_0^2 - 1)\gamma = 0 \tag{3.14}$$

In conformity with the assumption about the piecewise constant pressure we assume

$$q(z) = \begin{cases} 0, & -1 < z < z_0 \\ 1 & z_0 < z < 1 \end{cases}$$

Dividing (3.13) by (3.12), we obtain

$$[\alpha_1 + 3\alpha_2\gamma + \alpha_3\gamma^2 + \alpha_2\gamma^3]\{[\alpha_1 - 2\cdot 231^{-1}(1-\nu^2)]\gamma^3 + \tag{3.15}$$
$$3\alpha_2\gamma^2 + \alpha_3\gamma + \alpha_2\}^{-1} = 0.485\,(1 + z_0)^2\,(1 + z_0 / 2)^{-1}$$

$$\alpha_1 = (2\eta - 1)^2 - \left(\frac{1-\nu}{2}\right)^2 - \frac{3}{14}(1 - \nu^2), \quad \alpha_2 = (1+\nu)(2\eta - 1) / \sqrt{15}$$

$$\alpha_3 = (2\eta - 1)^2 + \frac{2}{15}(1+\nu)^2 - \left(\frac{1-\nu}{2}\right)^2 - \frac{1-\nu^2}{14}$$

Equations (3.14) and (3.15) form a closed system of algebraic equations for the determination of γ and z_0. Having determined γ and z_0 we find s by using one of Eqs. (3.12). The results may be presented in the form

$$w_0 = n_1 b\left(\frac{G}{E\delta b}\right)^{1/3}, \quad N_1 = \frac{n_2 E\delta w_0^2}{b^2}, \quad N_2 = \frac{n_3 E\delta w_0^2}{b^2} \tag{3.16}$$
$$p_1 - p_a = n_4 \frac{G}{\pi b^2}$$

where w_0 is the maximum deflection of the diaphragm, N_1 and N_2 are tensions at the inner and outer boundaries of restraints, and n_1, n_2, n_3 and n_4 are dimensionless coefficients which depend on a / b. Values of these coefficients and of γ and z_0 are given in Table 1 for $\nu = 0.47$.

Table 1

a/b	n_1	n_2	n_3	n_4	z_0	γ
0.184	0.13	10.0	6.3	1.78	0.1	0.16
0.221	0.121	10.9	7.08	1.73	0.12	0.195
0.26	0.113	11.6	7.7	1.67	0.14	0.23
0.343	0.097	14.2	9.95	1.55	0.2	0.352

4. To check the validity of assumptions made in the derivation of formulas (2.3) and (3.16) experiments were carried out with aerostatic supports. The range of loads was 160—1800 kg and the air flow rate was 0.2—0.56 m^3 / min. The static pressure p and the clearance h between the Dural base plate and the 3 mm thick rubber diaphragm were measured in the neighborhood of the minimum gap.

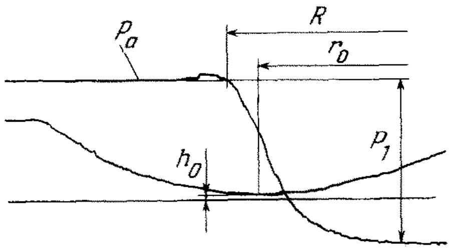

Fig. 5

Since at some modes the minimum gap did not exceed a few hundredth of a millimeter, measurement of h had presented considerable difficulties. None of the methods of small gap measurement described in [4] could be used. The mechanical contact of a mobile needle with the diaphragm was used. The needle was supported by a spring held on a bracket attached to the underside of the mobile top of the stand, and could freely move in the vertical direction. The sensing element of a strain gauge was glued to the spring. The deformation of the spring by the contact of the needle with the diaphragm was transferred to an oscillogram via the strain gauge, amplifier and oscillograph.

A typical oscillogram is shown in Fig. 5.

It is seen from the oscillogram that in the radial direction the length of the section along which the pressure drop (from p_1 inside the cushion to the atmospheric pressure p_a) is equal 4mm. This length varies considerably with the change of experimental conditions: it decreases with increasing load and increases with increasing flow rate. The radius r_0 of the minimum gap is smaller than radius R at which the pressure drops to atmospheric. For practical calculations it can be assumed that $R = r_0$. Under conditions of higher air flow rates a zone of rarefaction at the exit of air from the minimum gap is observed. This is explained by the inertia of the flowing air. The length of such zone is 2 — 2.5 mm. This justifies the assumptions made in Sect. 2 about incompressibility and the small effect of the inertia term in Eq. (2.3).

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